

# Sum rules for hadronic contributions to low-energy light-by-light scattering

Vladimir Pascalutsa, Vladyslav Pauk, Marc Vanderhaeghen

Institut für Kernphysik, Johannes Gutenberg Universität, Mainz D-55099, Germany

E-mail: vladipas@kph.uni-mainz.de

**Abstract.** We present a set of sum rules relating the low-energy light-by-light scattering to integrals of  $\gamma\gamma$  fusion cross-sections and use them to study the hadronic contributions.

## 1. Introduction

Some decades ago a general analysis of the forward Compton scattering amplitude allowed Baldin [1], Gerasimov [2], Drell and Hearn [3] to establish first sum rules expressing the static electromagnetic properties of the nucleon in terms of its total photoabsorption cross sections. The Baldin sum rule relates the sum of the electric  $\alpha_E$  and magnetic  $\beta_M$  polarizabilities to an integral of the unpolarized photoabsorption cross section  $\sigma$ :

$$\alpha_E + \beta_M = \frac{1}{2\pi^2} \int_0^\infty \frac{d\nu}{\nu^2} \sigma(\nu), \quad (1)$$

where  $\nu$  is the photon energy in the lab frame. The Gerasimov-Drell-Hearn (GDH) sum rule relates the anomalous magnetic moment  $\kappa$  of a spin-1/2 target to an integral of the helicity-difference photoabsorption cross section  $\Delta\sigma = \sigma_{3/2} - \sigma_{1/2}$  (here subscripts stand for the value of total helicity):

$$\frac{e^2}{2M^2} \kappa^2 = \frac{1}{\pi} \int_0^\infty \frac{d\nu}{\nu} \Delta\sigma(\nu), \quad (2)$$

with  $e$  the charge and  $M$  the mass of the target.

A few decades later it has been realized [4–6] that the GDH sum rule applies to a photon target, in which case the anomalous magnetic moment is zero by Furry’s theorem, and one simply has:

$$0 = \frac{1}{\pi} \int_0^\infty \frac{d\nu}{\nu} [\sigma_2(\nu) - \sigma_0(\nu)], \quad (3)$$

where  $\sigma_\lambda$  is the  $\gamma\gamma$ -fusion cross section with the total helicity  $\lambda$ , and  $\nu$  is the photon energy in collider kinematics.

More recently, a systematic derivation of sum rules for light-light system has been done [7], resulting in sum rules for the low-energy constants of the Euler-Heisenberg Lagrangian. Some details of the sum rules for light-by-light (LbL) scattering will be given below, with emphasis on perturbative verifications of the sum rules in field theory. The case of charged massive spin-1 field is especially interesting.

In the concluding section we shall discuss the way the sum rules can be used to evaluate the hadronic contributions to the low-energy interaction of light with light.

## 2. Sum rules for light-by-light scattering

The sum rules in question are based on very general properties of the S-matrix, namely Lorentz and crossing symmetries, analyticity, unitarity, as well the gauge symmetry of the electromagnetic interaction. To demonstrate this we highlight here the main derivation steps for the case of  $\gamma\gamma$  systems.

Denoting LbL scattering ( $\gamma\gamma \rightarrow \gamma\gamma$ ) corresponding Feynman and helicity amplitudes as, respectively,  $\mathcal{M}$  and  $M$ , their relation is:

$$M_{\lambda_1\lambda_2\lambda_3\lambda_4} = \varepsilon_{\lambda_4}^{*\mu_4}(\vec{q}_4) \varepsilon_{\lambda_3}^{*\mu_3}(\vec{q}_3) \varepsilon_{\lambda_2}^{\mu_2}(\vec{q}_2) \varepsilon_{\lambda_1}^{\mu_1}(\vec{q}_1) \times \mathcal{M}_{\mu_1\mu_2\mu_3\mu_4}, \quad (4)$$

where  $\varepsilon(\vec{q})$  are the photon polarization 4-vectors,  $\lambda$ 's are the helicities; for real photons traveling along the  $z$  axis, i.e.  $\vec{q} = (0, 0, \nu)$ , the polarization vectors are  $\varepsilon_\lambda(\pm\vec{q}) = 2^{-1/2}(0, \mp\lambda, -i, 0)$ . The Mandelstam variables are defined as  $s = (q_1 + q_2)^2 = 4\nu^2$ ,  $t = (q_1 - q_3)^2$ ,  $u = (q_1 - q_4)^2$ , with  $q_i$  the photon 4-momenta.

In the forward kinematics, where  $q_3 = q_1$ ,  $q_4 = q_2$ , and hence  $t = 0$ ,  $u = -s$ , the general Lorentz structure of the Feynman amplitude is given by:

$$\begin{aligned} \mathcal{M}_{\mu_1\mu_2\mu_3\mu_4} &= A(s) g_{\mu_4\mu_2} g_{\mu_3\mu_1} + B(s) g_{\mu_4\mu_1} g_{\mu_3\mu_2} \\ &+ C(s) g_{\mu_4\mu_3} g_{\mu_2\mu_1}, \end{aligned} \quad (5)$$

where  $g_{\mu\nu}$  is the Minkowski metric. Crossing symmetry (under  $1 \leftrightarrow 3$ , or  $2 \leftrightarrow 4$ ) means in this case

$$\begin{aligned} \mathcal{M}_{\mu_1\mu_2\mu_3\mu_4} &= A(u) g_{\mu_4\mu_2} g_{\mu_3\mu_1} + B(u) g_{\mu_4\mu_3} g_{\mu_1\mu_2} \\ &+ C(u) g_{\mu_1\mu_4} g_{\mu_3\mu_2}, \end{aligned} \quad (6)$$

hence  $A(-s) = A(s)$ ,  $B(-s) = C(s)$ . As a result there are three independent nonvanishing helicity amplitudes:

$$\begin{aligned} M_{++++}(s) &= A(s) + C(s), \\ M_{+--+}(s) &= A(s) + B(s), \\ M_{++--}(s) &= B(s) + C(s), \end{aligned} \quad (7)$$

satisfying the following crossing relations:  $M_{+--+}(s) = M_{++++}(-s)$ , and  $M_{++--}(s) = M_{+--+}(-s)$ .

The principle of (micro-)causality implies that the above functions are analytic functions of  $s$  everywhere in the complex  $s$  plane except along the real axis. For the amplitudes,

$$f^{(\pm)}(s) = M_{++++}(s) \pm M_{+--+}(s), \quad (8a)$$

$$g(s) = M_{++--}(s), \quad (8b)$$

the analyticity infers the following dispersion relations:

$$\text{Re} \left\{ \frac{f^{(\pm)}(s)}{g(s)} \right\} = \frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{ds'}{s' - s} \text{Im} \left\{ \frac{f^{(\pm)}(s')}{g(s')} \right\}, \quad (9)$$

where  $f$  indicates the principal-value integration. These relations hold as long as the integral converges, and otherwise subtractions are needed. Because  $f^{(\pm)}(-s) = \pm f^{(\pm)}(s)$  and  $g(-s) = g(s)$ , we can express the right-hand side as an integral over positive  $s$  only:

$$\text{Re} \left\{ \begin{array}{c} f^{(+)}(s) \\ g(s) \end{array} \right\} = \frac{2}{\pi} \oint_0^\infty \frac{ds' s'}{s'^2 - s^2} \text{Im} \left\{ \begin{array}{c} f^{(+)}(s') \\ g(s') \end{array} \right\}, \quad (10a)$$

$$\text{Re} f^{(-)}(s) = -\frac{2s}{\pi} \oint_0^\infty ds' \frac{\text{Im} f^{(-)}(s')}{s'^2 - s^2}. \quad (10b)$$

In the physical region ( $s \geq 0$ ), the optical theorem relates the imaginary part of these amplitudes to the total absorption cross-sections with definite polarization of the initial  $\gamma\gamma$  state:

$$\text{Im} f^{(\pm)}(s) = -\frac{s}{8} [\sigma_0(s) \pm \sigma_2(s)], \quad (11a)$$

$$\text{Im} g(s) = -\frac{s}{8} [\sigma_{||}(s) - \sigma_{\perp}(s)]. \quad (11b)$$

Substituting these expressions in the above dispersion relations one obtains:

$$\text{Re} f^{(+)}(s) = -\frac{1}{2\pi} \oint_0^\infty ds' s'^2 \frac{\sigma(s')}{s'^2 - s^2}, \quad (12a)$$

$$\text{Re} f^{(-)}(s) = -\frac{s}{4\pi} \oint_0^\infty ds' \frac{s' \Delta\sigma(s')}{s'^2 - s^2}, \quad (12b)$$

$$\text{Re} g(s) = -\frac{1}{4\pi} \oint_0^\infty ds' s'^2 \frac{\sigma_{||}(s') - \sigma_{\perp}(s')}{s'^2 - s^2}, \quad (12c)$$

where  $\sigma = (\sigma_0 + \sigma_2)/2 = (\sigma_{||} + \sigma_{\perp})/2$  is the unpolarized total cross section, and  $\Delta\sigma = \sigma_2 - \sigma_0$  (0 or 2 show the total helicity of the circularly polarized photons, while  $||$  or  $\perp$  show if the linear photon polarizations are parallel or perpendicular).

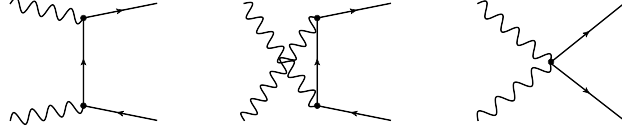
We next recall that gauge invariance and discrete symmetries constrain the low-energy photon-photon interaction to the Euler-Heisenberg form [8], given by the following Lagrangian density:

$$\mathcal{L}_{\text{EH}} = c_1 (F_{\mu\nu} F^{\mu\nu})^2 + c_2 (F_{\mu\nu} \tilde{F}^{\mu\nu})^2, \quad (13)$$

where  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ ,  $\tilde{F}^{\mu\nu} = \varepsilon^{\mu\nu\alpha\beta} \partial_\alpha A_\beta$ . Expanding the left-hand side and right-hand side of Eq. (12) in powers of  $s$  and matching them at each order yields a number of sum rules. At 0th order in  $s$  we would find

$$0 = \int_0^\infty ds [\sigma_{||}(s) \pm \sigma_{\perp}(s)], \quad (14)$$

which cannot work for “+” since the unpolarized cross-section is a positive-definite quantity. Empirically  $\sigma$  shows a slowly rising behavior at large  $s$  and the integral diverges. The assumption of an unsubtracted dispersion relation is violated in this case. For the “−” case the sum rule is broken too, cf. [5] and references therein.



**Figure 1.** Pair production in scalar QED.

At the first and second orders we find, respectively:

$$0 = \int_0^\infty ds \frac{\Delta\sigma(s)}{s}, \quad (15a)$$

$$c_1 \pm c_2 = \frac{1}{8\pi} \int_0^\infty ds \frac{\sigma_{||}(s) \pm \sigma_{\perp}(s)}{s^2}. \quad (15b)$$

The first sum rule here is the analog of the GDH sum rule mentioned above, while the sum rules for the low-energy constants are unique to the  $\gamma\gamma$  system. Note that according to these sum rules the constants  $c_1$  and  $c_2$  are positive definite, in contrast to some previous predictions in the literature [9].

Most of the above arguments will equally hold for the space-like virtual photons ( $q_1^2 < 0$ ,  $q_2^2 < 0$ ), if written in a variable which reflects under crossing, e.g.  $\nu = s - q_1^2 - q_2^2$ . The EH Lagrangian must however be extended by terms containing  $\partial_\mu F^{\mu\nu}$ . For the case when at least one is real such terms are absent and the above sum rules hold with only a single modification:  $s \rightarrow \nu = s - q^2$ , where  $q^2$  is the other photon virtuality.

### 3. Verifications in perturbative QED

The leading-order cross-sections of  $\gamma\gamma$ -fusion in QED are given by tree-level pair-production diagrams (Fig. 1), which can be easily computed and substituted into the right-hand-side of the sum rules in Eq. (15). In QED of a scalar and a spinor (Dirac) particle the superconvergence sum rule has thus been verified to leading order the fine-structure constant  $\alpha$ . For example the spin-1/2 pair-production helicity-difference cross section in  $\gamma^*\gamma$  collision is given by

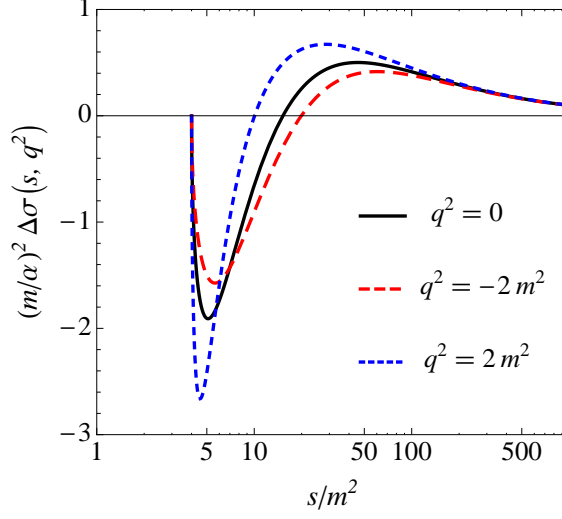
$$\begin{aligned} \Delta\sigma^{(\gamma^*\gamma \rightarrow f\bar{f})}(s, q^2, 0) &= \frac{8\pi\alpha^2}{(s - q^2)^2} \theta(s - 4m^2) \\ &\times \left\{ -(3s + q^2) \sqrt{1 - \frac{4m^2}{s}} + 2(s + q^2) \operatorname{arctanh} \sqrt{1 - \frac{4m^2}{s}} \right\}, \end{aligned} \quad (16)$$

and is plotted in Fig. 2 for three different values of  $q^2$ . One sees that in all cases the low- and high-energy contributions cancel. The fact that

$$\int_{4m^2}^\infty ds \frac{\Delta\sigma^{(\gamma^*\gamma \rightarrow f\bar{f})}(s, q^2, 0)}{s - q^2} = 0 \quad (17)$$

is easily verified for any  $q^2 < 4m^2$ .

Substituting the corresponding linearly-polarized cross section into the sum rule for EH low-energy constants, we obtain:  $c_1 = \frac{7\alpha^2}{1440m^4}$ ,  $c_2 = \frac{\alpha^2}{1440m^4}$  for the scalar case;  $c_1 = \frac{\alpha^2}{90m^4}$ ,  $c_2 = \frac{7\alpha^2}{360m^4}$  for the spinor case. This result agrees with the explicit one-loop calculations of low-energy light-by-light scattering, see e.g. [10, 11].



**Figure 2.** Helicity-difference cross section of  $\gamma^*\gamma \rightarrow f\bar{f}$  in QED at leading order, for different photon virtualities.

The case of spin-1 QED, describing a charged and massive vector particle, is not as simple. In this case interaction with electromagnetic field can be described in terms of three independent structure functions. Considering the minimal and linear non-minimal couplings we start with the following Lagrangian density:

$$\begin{aligned} \mathcal{L}_1 = & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}W_{\mu\nu}^*W^{\mu\nu} - M^2W_\mu^*W^\mu + ieW_{\mu\nu}^*A^\mu W^\nu - ieA_\mu W_\nu^*W^{\mu\nu} + e^2A^2W_\mu^*W^\mu + \\ & +iel_1W_\mu^*W_\nu F^{\mu\nu} + el_2[(D_\mu^*W_\nu^*)W^\alpha\partial_\alpha F^{\mu\nu} + W_\alpha^*(D_\mu W_\nu)\partial^\alpha F^{\mu\nu}]/(2M^2), \end{aligned} \quad (18)$$

where  $A_\mu$  and  $W_\mu$  denote the electromagnetic and vector-boson fields respectively;  $F_{\mu\nu} = D_\mu F_\nu - D_\nu F_\mu$  and  $W_{\mu\nu} = D_\mu W_\nu - D_\nu W_\mu$  are the corresponding field-strength tensors, with  $D_\mu = \partial_\mu - ieA_\mu$  the covariant derivative,  $e$  the charge and  $M$  the mass of the vector boson. The parameters  $l_1$  and  $l_2$  contribute to the magnetic and quadrupole moments as:

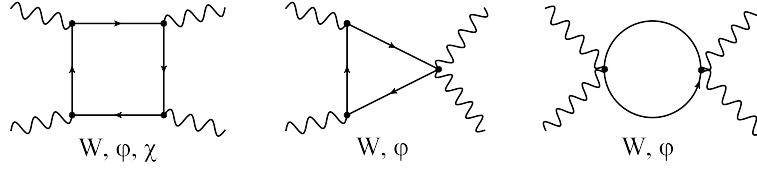
$$\mu = (1 + l_1)\frac{e}{2M} \quad \text{and} \quad Q = (l_2 - l_1)\frac{e}{M^2}. \quad (19)$$

Computing the cross-section for the tree-level  $\gamma\gamma \rightarrow W^+W^-$  process in this theory we find that the integral on the right-hand-side of the first (super-convergence) sum rule in Eq. (15) diverges, unless  $l_1 = 1$  and  $l_2 = 0$ . Only for the latter choice of the parameters the integral converges and is equal to 0, as it should. This choice of parameters is realized in the Standard Model, and gives rise to the so-called "natural values" of the electromagnetic moments.

Computing for the linearly polarized cross sections (for  $l_1 = 1$  and  $l_2 = 0$ ) and using the second sum rule we obtain:

$$c_1 = \frac{29\alpha^2}{160M^4}, \quad c_2 = \frac{27\alpha^2}{160M^4}. \quad (20)$$

This is in agreement with the one-loop electroweak correction to LbL scattering (Fig. 3), which we obtained by the low-energy expansion of the expressions of Bohm and Schuster [12]. It is interesting that to obtain this result in electroweak theory one needs to take care of the Higgs mechanism as well as the ghosts, while on the side of the sum rule the calculation is much simpler: tree-level production of massive vector bosons, and a dispersion integral.



**Figure 3.** One-loop Feynman graphs in the standard electroweak theory. Here  $W$  stands for the gauge bosons,  $\phi$  - Higgs fields and  $\chi$  denotes Fadeev-Popov ghosts.

#### 4. Hadronic contributions to the LbL scattering

While the role of the SRs in QED becomes fairly clear in these perturbative calculations, in quantum chromodynamics (QCD), in its non-perturbative regime, it is far less obvious. We can gain some insight by looking at individual contributions to the  $\gamma\gamma \rightarrow \text{hadrons}$  cross sections.

##### 4.1. Superconvergence sum rule

The high-energy behavior of  $\Delta\sigma$  is determined by a  $t$ -channel exchange of unnatural parity and is expected from Regge theory – in the absence of fixed pole singularities – to drop as  $1/s$  or faster [13]. We therefore expect the sum rule (15a) to converge. The dominant features of the  $\gamma\gamma$  to multihadron production comes firstly from the Born terms in the  $\pi^+\pi^-$  (or  $K^+K^-$ ) channels, which each separately obeys the sum rule. The largest contributions in the hadronic sector are thus expected to come from the resonance production:  $\gamma\gamma \rightarrow M$ , with  $M$  being a meson. It is highly nontrivial to see how the sum rule is saturated in this case.

As two real photons do not couple to a  $J^P = 1^-$  or  $1^+$  state due to the Landau-Yang theorem, one expects the dominant contribution to come from scalar, pseudoscalar, and tensor mesons. One can express the  $\gamma\gamma \rightarrow M$  cross section for a meson with spin  $J$ , mass  $m_M$ , and total width  $\Gamma_{tot}$ , using a Breit-Wigner parametrization, in terms of the decay width  $\Gamma_{\gamma\gamma}^{(\Lambda)}$  of the meson into two photons of total helicity  $\Lambda = 0, 2$ , as

$$\begin{aligned} \sigma_{\Lambda}^{\gamma\gamma \rightarrow M}(s) &= (2J+1) 16\pi \frac{\Gamma_{\gamma\gamma}^{(\Lambda)} \Gamma_{tot}}{(s - m_M^2)^2 + \Gamma_{tot}^2 m_M^2} \\ &\approx (2J+1) 16\pi^2 \frac{\Gamma_{\gamma\gamma}^{(\Lambda)}}{m_M} \delta(s - m_M^2), \end{aligned} \quad (21)$$

where the last line is obtained in the narrow resonance approximation. For the pseudoscalar mesons, which can only contribute to the helicity-zero cross section, the narrow resonance approximation is very accurate and allows to quantify their contribution as shown in Table 1. For the pion, this value is entirely driven by the chiral anomaly, which allows the expression of the  $\pi^0$  contribution to the sum rule as  $-\alpha^2/(4\pi f_\pi^2)$ , with  $f_\pi = 92.4$  MeV the pion decay constant.

To compensate the large negative contribution to the sum rule from pseudoscalar mesons, one needs to have an equivalent strength in the helicity-two cross section,  $\sigma_2$ . The dominant feature of the helicity-two cross section in the resonance region arises from the multiplet of tensor mesons  $f_2(1270)$ ,  $a_2(1320)$ , and  $f'_2(1525)$ . Measurements at various  $e^+e^-$  colliders, notably recent high statistics measurements by the BELLE Collaboration of the  $\gamma\gamma$  cross sections to  $\pi^+\pi^-$  [14],  $\pi^0\pi^0$  [15],  $\eta\pi^0$  [16], and  $K^+K^-$  [17] channels have allowed accurate confirmation of their parameters. As these tensor mesons were also found to be relatively well described by Breit-Wigner resonances, we use Eq. (21) to provide a first estimate of their contribution to the sum rule. We show the results in Table 2, both in the narrow width approximation and using a Breit-Wigner shape, assuming that the tensor mesons pre-dominantly contribute to  $\sigma_2$ , as is found by the above-mentioned experimental analyses of decay angular distributions.

	$m_M$ [MeV]	$\Gamma_{\gamma\gamma}$ [keV]	$\int ds \Delta\sigma/s$ [nb]
$\pi^0$	134.98	$(7.8 \pm 0.6) \times 10^{-3}$	$-195.0 \pm 15.0$
$\eta$	547.85	$0.51 \pm 0.03$	$-190.7 \pm 11.2$
$\eta'$	957.66	$4.30 \pm 0.15$	$-301.0 \pm 10.5$
Sum $\eta, \eta'$			$-492 \pm 22$

**Table 1.** Sum rule contribution of the lightest pseudoscalar mesons (last column). The experimental values of meson masses  $m_M$  and  $2\gamma$  decay widths  $\Gamma_{\gamma\gamma}$  are from PDG [18].

	$m_M$ [MeV]	$\Gamma_{\gamma\gamma}$ [keV]	$\int ds \Delta\sigma/s$ narrow res. [nb]	$\int ds \Delta\sigma/s$ Breit-Wigner [nb]
$a_2(1320)$	1318.3	$1.00 \pm 0.06$	$134 \pm 8$	$137 \pm 8$
$f_2(1270)$	1275.1	$3.03 \pm 0.35$	$448 \pm 52$	$479 \pm 56$
$f'_2(1525)$	1525	$0.081 \pm 0.009$	$7 \pm 1$	$7 \pm 1$
Sum $f_2, f'_2$			$455 \pm 53$	$486 \pm 57$

**Table 2.** Sum rule contribution of the lowest tensor mesons. We show both results in the narrow resonance approximation (4th column) and using a Breit-Wigner parametrization (last column). The experimental values of meson masses  $m_M$  and  $2\gamma$  decay widths  $\Gamma_{\gamma\gamma}$  are from PDG [18].

Comparing Tables 1 and 2, we can see that the contribution of the lowest isovector tensor meson composed of light quarks,  $a_2(1320)$ , compensates to around 70 % the contribution of the  $\pi^0$ , which is entirely governed by the chiral anomaly. For the isoscalar states composed of light quarks, the cancellation is even more remarkable, as the sum of  $f_2(1270)$  and  $f'_2(1525)$  cancels entirely, within the experimental accuracy, the combined contribution of the  $\eta$  and  $\eta'$ .

Besides the tensor mesons, the subdominant resonance contributions to the  $\gamma\gamma$  total cross section arise from the scalar mesons  $f_0/\sigma(600)$ ,  $f_0(980)$ , and  $a_0(980)$ . A reliable estimate of the scalar mesons requires an amplitude analysis of the partial channels, see e.g. [19]. A future study will estimate more precisely the scalar meson helicity-zero contribution to the sum rule, and elaborate on the cancellation between the tensor mesons and the (pseudo)scalar meson contributions in the sum rule of Eq. (15a). Interestingly, when going to the charm sector, the sum rule also implies a cancellation between the  $\eta_c$  meson, whose contributions amounts to about  $-15.5$  nb, and scalar and tensor  $c\bar{c}$  states.

In the case of non-zero virtuality of one of the photons, the cross-section of the process  $\gamma^*\gamma \rightarrow M$  is defined not by a Breit-Wigner approximation, but rather by a function of  $Q^2$ , so-called transition form-factor (TFF). Application of the superconvergent sum rule to these processes can give a useful information of the electromagnetic structure of mesons. Feynman amplitude of the process  $\gamma^*\gamma \rightarrow M$  for the case of pseudoscalar meson has the form:

$$T_{\mu\nu} = ie^2 \varepsilon_{\mu\nu\alpha\beta} q^\alpha q'^\beta F_{\pi^0\gamma^*\gamma}(Q^2), \quad (22)$$

where  $Q^2 = -q'^2$  denotes virtuality of the  $\gamma^*$  and  $F_{\pi^0\gamma^*\gamma}$  is the  $\gamma^*\gamma \rightarrow \pi^0$  TFF.

The leading contribution of the single  $\pi^0$  production to the superconvergent sum rule is found as:

$$\int_0^\infty ds \frac{\sigma_0(s)}{s + Q^2} = (4\pi\alpha)^2 \pi \left( \frac{1}{2} F_{\pi^0\gamma^*\gamma}(Q^2) \right)^2. \quad (23)$$

The production of a tensor meson contributes both to the helicity-0 and helicity-2 amplitudes. However the contribution of helicity-0 cross-section is usually negligible, while the helicity-2 contribution is given by

$$\int_0^\infty ds \frac{\sigma_2(s)}{s+Q^2} = (4\pi\alpha)^2 \pi \left( F_{a_2\gamma^*\gamma}^{\Lambda=2}(Q^2) \right)^2 + \mathcal{O}\left(\frac{1}{Q^2}\right). \quad (24)$$

Thus, assuming that in the isovector channel the sum rule is saturated by  $\pi^0$  ( $\Lambda = 0$ ) and  $a_2$  ( $\Lambda = 2$ ) mesons, we obtain at low  $Q^2$  the following relation between the TFFs:

$$F_{a_2\gamma^*\gamma}^{\Lambda=2}(Q^2) \approx \frac{1}{2} F_{\pi^0\gamma^*\gamma}(Q^2), \quad (25)$$

which, as seen from Tables 1 and 2, is at  $Q^2 = 0$  satisfied to an accuracy of better than 70%.

#### 4.2. The low-energy-constant sum rules

The sum rules

$$c_1 = \frac{1}{8\pi} \int_0^\infty ds \frac{\sigma_{||}(s)}{s^2}, \quad (26)$$

$$c_2 = \frac{1}{8\pi} \int_0^\infty ds \frac{\sigma_{\perp}(s)}{s^2}, \quad (27)$$

allow one to assess the size of the hadronic contribution to the low-energy LbL scattering. For example, the pseudo-scalar meson production in  $\gamma\gamma^*$  fusion yields:

$$\begin{aligned} \sigma_{||} &= 0, \\ \sigma_{\perp} &= 4\pi^3 \alpha^2 |F_{M\gamma\gamma^*}(Q^2)|^2 (s+Q^2) \delta(s-m_M^2), \end{aligned} \quad (28)$$

which lead to  $c_1 = 0$  and

$$c_2 = \frac{2\pi F_{\gamma\gamma}}{m_M^5}. \quad (29)$$

The corresponding numerical results are given in Table 3. This result, however, does not aid much in the problem of hadronic LbL contributions to muon anomaly,  $(g-2)_\mu$ , where the main effect comes from the LbL scattering at the hadronic scale, see e.g. [20].

	$c_1$	$c_2$ [ $10^{-4} \text{ GeV}^{-4}$ ]
$\pi^0$	0	10.8
$\eta$	0	0.7
$\eta'$	0	0.4

**Table 3.** Contribution of the light pseudoscalar mesons to low-energy LbL scattering.



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